1 The existence of an equivalent martingale measure for all stopped processes does not implies the existence of an absolute continuous martingale measure for the whole process even in the dyadic case

We fix a probability space $(\Omega, \mathcal{F}, I\!P)$. We take $(\varepsilon_i,\ i \geq 1)$ to be independent and identically distributed with $I\!P(\varepsilon_1 = +1) = I\!P(\varepsilon_1 = -1) = 1/2$. We define the filtration

$$\mathcal{F}_t := \sigma(\varepsilon_1, \ldots, \varepsilon_t).$$

We consider a financial model with two assets, the price process S of the risky asset is given by

$$S_t := s_0 + \sum_{i=1}^{t} (\varepsilon_i + \alpha_i),$$

where s_0 and $-1 < \alpha_i < 1$ real constants. For the moment we take the bond price identically 1.

W. Schachermayer asked the following question in this context:

Is it true that if for all stopping time T satisfying $I\!P(T < \infty) = 1$ there exists an equivalent martingale measure on $\mathcal{F}_T$ for the stopped process S^T then there exists a martingale measure $Q << I\!P$ for the whole process ?

We shall show that the answer to this question is no whenever $\sum_{i=1}^{\infty} \alpha_i^2 = \infty$.

First we prove that, for any $t \in I\!N$ there exists one and only one measure Q_t defined on $\mathcal{F}_t$ such that $S_i,\ i \leq t$ is a Q_t-martingale. Furthermore $Q_t \sim I\!P$.

We set

$$\frac{dQ_t}{dI\!P}(\varepsilon_i = \eta_i,\ 1 \leq i \leq t) = \prod_{i=1}^{t} (1 - \eta_i \alpha_i). \tag{1.1}$$

Using independence, one verifies directly that this is an (equivalent) martingale measure up to time t. Uniqueness follows by induction: it is trivial for $t = 1$, let us suppose that the assertion is true up to time t, we would like to show it for $t + 1$, assuming that the only martingale measure on $\mathcal{F}_t$ is Q_t. We take any martingale measure N up to time $t + 1$. We necessarily have

$$N(\varepsilon_{t+1}=\eta|\mathcal{F}_t)=\frac{1-\eta\alpha_{t+1}}{2}.$$

Taking any set $A\in\mathcal{F}_{t+1}$ there is $\eta\in\{+1,-1\}$ and $B\in\mathcal{F}_t$ such that $A=B\cap\{\varepsilon_{t+1}=\eta\}$. This proves $N(A)=Q_{t+1}(A)$.

The family Q_t is compatible so it defines a unique martingale measure on $\sigma(\varepsilon_i,\ i\geq 1)$. We denote this measure by Q. Using Kakutani's theorem we infer that Q and $I\!P$ are mutually singular provided that $\sum_{i=1}^{\infty}\alpha_i^2=\infty$. So there is no absolutely continuous martingale measure in this case.

We now show that there is an *equivalent* martingale measure on $\mathcal{F}_T$ for the stopped process S^T, where T is an almost surely finite stopping time. The restrictions of Q on $\mathcal{F}_t$ defined by (1.1) are equivalent to $I\!P$. By Theorem 2.36 on p. 217 of [7] $Q|\mathcal{F}_T\sim I\!P|\mathcal{F}_T$ holds iff

$$\sum_{i=1}^{T}(\sqrt{1+\alpha_i}+\sqrt{1-\alpha_i})\ <\ \infty$$

and

$$\sum_{i=1}^{T}\left(\frac{1}{\sqrt{1+\alpha_i}}+\frac{1}{\sqrt{1-\alpha_i}}\right)\ <\ \infty.$$

It is easy to check (using Taylor-series expansion) that both conditions are equivalent to

$$\sum_{i=1}^{T}\alpha_i^2<\infty,$$

which is trivially true for $a.s.$ finite T.

2 The set of equivalent martingale measures is not $\sigma(L^1,L^\infty)$ compact

In this section we consider a one-step market model with one asset. We demonstrate that even in this simple case the set of martingale measures fails to be $\sigma(L^1,L^\infty)$ compact.

We suppose that the return R of some risky asset between time 0 and 1 is uniformly distributed on $[-1,1]$. The density $dQ/dI\!P$ provides a martingale measure if and only if

$$E\frac{dQ}{dI\!P}R=0.$$

We may confine our attention to $dQ/dI\!P$ which are $\sigma(R)$-measurable, hence $dQ/dI\!P = \phi(R)$ for some measurable $\phi : [-1,1] \to I\!R$. The family of such ϕ is characterized by

$$\begin{aligned} \phi &\geq 0, \\ \int_{[-1,1]} \phi(x) x \, dx &= 0, \\ \frac{1}{2} \int_{[-1,1]} \phi(x) dx &= 1. \end{aligned} \tag{2.2}$$

We only regard the ϕ which satisfy

$$\phi(x) = \phi(-x), \ x \in [-1,1].$$

We shall construct a sequence ϕ_n satisfying (2.2) such that

$$\lim_{n\to\infty} E\phi_n(R) I_{\phi_n(R)\geq n} = \lim_{n\to\infty} \frac{1}{2} \int_{[-1,1]} \phi_n(x) I_{\phi_n(x)\geq n} dx > 0.$$

Such a family is not uniformly integrable, hence the set of martingale measures (which contains the above sequence) is not relatively compact in the topology $\sigma(L^1, L^\infty)$ (see [6]).

We define ϕ_n as follows:

$$\phi_n(x) := n, \ x \in (-\frac{1}{4n}, \frac{1}{4n}), \ \phi_n(x) := \frac{3n}{4n-1}, \ x \in [-1,1] \setminus (-\frac{1}{4n}, \frac{1}{4n}).$$

One can easily check that this sequence satisfies (2.2) and that

$$E\phi_n(R) I_{\phi_n(R)\geq n} = \frac{1}{4}.$$

3 No arbitrage in each finite submarket does not suffice for the existence of an absolutely continuous martingale measure in the large market

We consider a one-period market model with countably many assets, whoses returns $R^i := \varepsilon_i - b_i \ i \geq 1$. We suppose that the ε_i are independent and the support of their distribution is $[-M, M]$ for some $M > \frac{1}{2}$. We furthermore assume $E\varepsilon_i = 0$ and $E\varepsilon_i^2 = 1$.

This is a fairly simple example of a large financial markets taken from [3]. It is still unknown, however, what is the necessary and sufficient condition on the b_i which guarantees the existence of an equivalent martingale measure, i.e. $Q \sim \mathbb{P}$ such that

$$E^Q R^i = 0,\ i \geq 1.$$

Now we sketch an example where absence of arbitrage holds in each finite submarket, however there is no martingale measure $Q << \mathbb{P}$ for the whole market.

We assert that if each b_i lies in $(-M, M)$ then (NA) holds (i.e. there is no arbitrage) on finite submarkets. By Dalang-Morton-Willinger Theorem this is equivalent to the existence of $Q_n \sim \mathbb{P}$ for $n \in \mathbb{N}$ such that

$$E^{Q_n} R^i = 0,\ 1 \leq i \leq n.$$

By independence, it suffices to produce such a measure for a fixed asset. One can give a direct construction of the density, we refer to [5].

If there is a martingale measure $Q << \mathbb{P}$ for the whole market, since

$$b_i = E\frac{dQ}{d\mathbb{P}}\varepsilon_i,$$

are just the Fourier-coefficients of the function $dQ/dP \in L^1$ with respect to the uniformly bounded orthonormed system ε_i, a theorem of Mercer applies (see p. 66 of [1]) and

$$b_i \to 0,\ i \to \infty.$$

Putting together our observations so far: if we set $b_i = 1/2,\ i \in \mathbb{N}$ we have that there is (NA), however no martingale measure $Q << \mathbb{P}$ exists as

$$b_i \to 0$$

obviously fails.

It is possible to show in this case that there is a "free lunch" (in fact, one can show in a similar way that there is a free lunch "of the second kind", in the sense of [3]). We take a porfolio ϕ^n where $\phi_i^n,\ 1 \leq n$ represents the amount of asset i we have. We set

$$\phi_i^n := -1/n.$$

It is easy to check that the value $V_n = -\frac{1}{n}\sum_{i=1}^{n}(\varepsilon_i - 1/2)$ of the portfolio ϕ^n converges to $1/2$ almost surely, and V_n^- remains bounded.

If b_i tends to 0 (slowly) then it is still possible to produce a free lunch in a certain sens, but in this case we can not guarantee that the negative parts are bounded. We take

$$b_i := \frac{1}{1 + \ln i}$$

and portfolios $\phi_i^n := -\frac{1}{n^{3/4}}$. The central limit theorem shows that

$$-\frac{\sum_{i=1}^{n} \varepsilon_i}{n^{3/4}} \to 0$$

in probability. The sum $\sum_{i=1}^{n} b_i$ is of the same order as

$$\int_1^n \frac{dx}{\ln x} = \int_0^{\ln n} \frac{e^t}{t} dt.$$

We find that for t large enough

$$\frac{e^t}{t} \geq e^{\frac{4}{5}t},$$

hence

$$\frac{\sum_{i=1}^{n} b_i}{n^{3/4}} \geq C + \frac{5}{4} n^{\frac{1}{20}}.$$

so the values $V_n = -\frac{1}{n^{3/4}} \sum_{i=1}^{n} (\varepsilon_i - b_i)$ tend to ∞ in probability.

4 L^1 theory is different from L^r theory for $r > 1$

We present a version of the dyadic model when the underlying probability space is $\Omega := (0, 1]$ considered with its Borel field and the Lebesgue measure, called here $I\!P$. We suppose this time that the ε_i are the Rademacher system, that is to say,

$$\varepsilon_i(x) = (-1)^k, \quad x \in \left(\frac{k}{2^i}, \frac{k+1}{2^i}\right], \ 0 \leq k \leq 2^i - 1.$$

We remark that the e_i are independent and identically distributed,

$$I\!P(\varepsilon_i = +1) = I\!P(\varepsilon_i = -1) = 1/2.$$

It is known that if $-1 < b_i < 1$ then

$$\sum_{i=1}^{\infty} b_i^2 < \infty$$

is equivalent to the existence of a martingale measure $Q \sim I\!P$ for the whole market (see [4]). We now construct a sequence b_i such that there is a martingale measure $Q \sim I\!P$ with density in $L^1 \setminus L^r$ but

$$\sum_{i=1}^{\infty} b_i^r = \infty,$$

where $r > 1$.

We define

$$f(x) = \frac{2^n S}{n^{1+\frac{1}{r}}},\ x \in (1/2^n, 1/2^{n-1}],\ n \geq 1,$$

where

$$S := \sum_{i=1}^{\infty} \frac{1}{n^{1+\frac{1}{r}}}.$$

This function satisfies $Ef = 1$ so we set

$$dQ/dI\!P := f.$$

On the other hand, $f \notin L^r$ for $r > 1$:

$$Ef^r = S^r \sum_{n=1}^{\infty} \frac{2^{n(r-1)}}{n^{r+1}} = \infty.$$

We define b_i by

$$b_i := \int_{(0,1]} f \varepsilon_i.$$

First we remark that by the symmetric nature of the Rademacher system

$$\int_{(\frac{1}{2^{i-1}},1]} f \varepsilon_i = 0.$$

So we get

$$\begin{aligned} b_i &= \int_{(0,\frac{1}{2^{i-1}}]} f\varepsilon_i = \sum_{j=i+1}^{\infty} \int_{(\frac{1}{2^j},\frac{2}{2^j}]} f\varepsilon_j + \int_{(\frac{1}{2^i},\frac{2}{2^i}]} f\varepsilon_i \\ &= S \sum_{j=i+1}^{\infty} \frac{1}{j^{1+\frac{1}{r}}} - \frac{S}{i^{1+\frac{1}{r}}}, \end{aligned}$$

since $\varepsilon_i = -1$ on $(\frac{1}{2^j}, \frac{2}{2^j}]$ and $\varepsilon_i = +1$ on $\cup_{j=i+1}^{\infty}(\frac{1}{2^j}, \frac{2}{2^j}]$.

The sum $\displaystyle\sum_{j=i+1}^{\infty} \frac{1}{j^{1+\frac{1}{r}}}$ has the same the behaviour as

$$\int_{i+1}^{\infty} \frac{dx}{x^{1+\frac{1}{r}}} = \frac{(i+1)^{-\frac{1}{r}}}{r}.$$

We infer that

$$b_i \approx -\frac{S}{i^{1+\frac{1}{r}}} + \frac{S}{ri^{\frac{1}{r}}} \approx C\frac{1}{i^{\frac{1}{r}}}.$$

This implies

$$\sum_{i=1}^{\infty} b_i^r = \infty$$

by

$$\sum_{i=1}^{\infty} 1/i = \infty.$$

We remark that we can even find b_i for each $\varepsilon > 0$ such that

$$\sum_{i=1}^{\infty} b_i^{\varepsilon} = \infty,$$

nevertheless there is a martingale measure (with density in $L^1 \setminus L^r$ for all $r > 1$).

5 Stationary markets

We fix a probability space $(\Omega, \mathcal{F}, I\!P)$. We consider a stationary process $(X_t)_{t\geq 1}$ with values in $I\!R$. Its filtration is denoted by $\mathcal{F}_t,\ t \geq 0$. We introduce investment opportunities Φ, such that Φ_t is $\mathcal{F}_t$-adapted and has a finite number of nonzero terms (i.e. $\Phi_t = 0$ if $t > T(\Phi)$). Thus there exists a sequence of measurable functions $(\varphi_t)_{t\geq 0}$, where $\varphi_0 \in I\!R$ and for $t \geq 1$ $\varphi_t : I\!R^t \to I\!R$ and

$$\Phi_t = \varphi_t(X_1, \ldots, X_t).$$

We will denote by $\mathcal{I}$ the set of investments. We take this set as a cone convex of $l^2(L^2)(\mathcal{F})$. We define the operator T from $l^2(L^2)(\mathcal{F})$ to $l^2(L^2)(\mathcal{F})$ by

$$T(\Phi) = (0, \varphi_0, \varphi_1(X_2), \ldots, \varphi_t(X_2, \ldots, X_{t+1}), \ldots).$$

We assume that the set $\mathcal{I}$ is stationary in the following sens :

$$T(\mathcal{I}) \subset \mathcal{I}.$$

We are interested in the caracterisation of the no-arbitrage condition in this context.

We say that there is no arbitrage up to time N if

$$\mathcal{I}_N \cap l^2(L^2_+) = \{0\},$$

where

$$\mathcal{I}_N := \{\Phi \in \mathcal{I} : \ T(\Phi) \leq N\}.$$

We can extend the single-ended stationary process to a doubled-end stationary process $(\tilde{X}_t)_{t\in\mathbb{Z}}$ such that $(X_t)_{t\geq 1}$ and $(\tilde{X}_t)_{t\geq 1}$ have the same distribution, see [2] p 105 proposition 6.5. The filtration of the new process will be denoted by $\tilde{\mathcal{F}}_t, \ t \in \mathbb{Z}$. We consider separators $H \in l^2(L^2)(\tilde{\mathcal{F}})$ i.e. of the form

$$H = \left(h_0(\tilde{X}_j, \ j \leq 0), h_1(\tilde{X}_j, \ j \leq 1), \ldots\right),$$

where $h_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ measurable functions.

We define the following sets of separators:

$$K_N := \{H \in l^2(L^2)_+(\tilde{\mathcal{F}}) :< H, \Phi >\leq 0, \ \forall \Phi \in \mathcal{I}_N, \ Eh_0(\tilde{X}_j, \ j \leq 0) = 1\}.$$

We define

$$K := \{H \in l^2(L^2)_+(\tilde{\mathcal{F}}) :< H, \Phi >\leq 0, \ \forall \Phi \in \mathcal{I}, \ Eh_0(\tilde{X}_j, \ j \leq 0) = 1\}.$$

We first remark that T can be extended to $\tilde{T}$ acting on elements of $l^2(L^2)(\tilde{\mathcal{F}})$.

$$\tilde{T}\Phi := \left(0, \varphi_0(\tilde{X}_j, \ j \leq 1), \varphi_1(\tilde{X}_j, \ j \leq 2), \ldots\right).$$

Now we consider the adjoint of $\tilde{T}$:

$$\tilde{T}^* H = \left(h_1(\tilde{X}_j, \ j \leq 0), h_2(\tilde{X}_j, \ j \leq 1), \ldots\right).$$

We claim that K is invariant under the action

$$\Psi(H) := \tilde{T}^* H / Eh_1(\tilde{X}_j, \ j \leq 0).$$

Indeed, we have for all $\Phi \in \mathcal{I}$:

$$< \Psi(H), \Phi > = < H, \tilde{T}\Phi / Eh_1(\tilde{X}_j,\ j \leq 0) > \leq 0,$$

because $\mathcal{I}$ is a cone invariant under the operator T, which coincides with $\tilde{T}$ here. We also notice that $\tilde{T}^*$ and hence Ψ are weakly continuous.

We need the following assumption which we will discuss in detail later:

Assumption. The set K is weakly compact in $l^2(L^2)(\tilde{\mathcal{F}})$.

Under this assumption the theorem of Tychonoff-Schauder provides a fixed point for Ψ. So we get $H \in l^2(L^2)(\tilde{\mathcal{F}})$ satisfying

$$\alpha h_t(\tilde{X}_j,\ j \leq t) = h_{t+1}(\tilde{X}_j,\ j \leq t); \quad t \geq 0. \tag{5.3}$$

where $\alpha := Eh_1(\tilde{X}_j,\ j \leq 0)$. This entails

$$h_{t+1}(\cdot) = \alpha h_t(\cdot)$$

almost surely $I\!P_{\{\tilde{X}_j,\, j \leq t\}}$, so by stationarity a.s. $I\!P_{\{\tilde{X}_j,\, j \leq 0\}}$. This leads to

$$\alpha^t h_0(\cdot) = h_t(\cdot)$$

almost surely $I\!P_{\{\tilde{X}_j,\, j \leq 0\}}$.

So we obtain a stationary deflator process $d_t := \alpha^t h_0(\tilde{X}_j,\ j \leq t)$ in $l^2(L^2)(\tilde{\mathcal{F}})$ such that for each $\Phi \in \mathcal{I}$

$$\sum_{i=0}^{T(\Phi)} Ed_t\Phi_t \leq 0.$$

Our main problem is to find conditions under which the above Assumption holds (maybe using other Banach spaces than $l^2(L^2)$). Even if the processus X_t has a finite state space, we find no way to infer $L^2(\tilde{\mathcal{F}})$-boundedness from $L^1(\tilde{\mathcal{F}})$-boundedness since we extend the single-ended process X_t to a double-ended process $\tilde{X}_t$ whose norm is given by an infinite sum.

We now give an example where we can show that the set in question is *not* compact. The construction is in the same spirit as section 2.

Example. We take X_i independent uniformly distributed on $[-M, M]$. We suppose the existence of an interest rate, i.e. an investment $(-1, 1+r)$. We also assume $M > 1+r$. In addition, we have one more investment

$$\Phi := (1, -X_1).$$

We find that

$$T\Phi = (0, 1, -X_2).$$

We look for separators H of a particular form:

$$H = \left(1, \frac{h(X_1)}{1+r}, \frac{h(X_2)}{(1+r)^2}, \dots\right).$$

We look for $h \geq 0$ such that

$$Eh(X_1) = 1, \quad Eh(X_1)X_1 \geq 1+r. \tag{5.4}$$

We look for functions $h_n : [-M, M] \to I\!\!R_+$ of the form:

$$\begin{aligned} h_n(x) &:= n, \;\; x \in (\alpha_n, M], \\ h_n(x) &:= c_n, \;\; x \in [-M, \alpha_n], \end{aligned}$$

with

$$\alpha_n := M - \frac{2(1+r)}{n-1}, \quad c_n := \frac{2M - n(M-\alpha)}{\alpha + M}.$$

We can easily check that for n sufficiently large c_n is non-negative and h_n fulfills the conditions (5.4). One can conclude that

$$Eh_n(X_1)I_{\{h_n(X_1)\geq n\}} = \frac{2(1+r)n}{n-1}.$$

As such a family is not uniformly integrable, the set K (which contains the above sequence) is not relatively compact in the topology $\sigma(L^1, L^\infty)$ (see [6]).

We would like to know if the assumption holds true only for finite Ω. We can relate this problem of campactness to certain reflexivity properties. Notably, if we take $\mathcal{I} \subset \ell^1(L^1)$ and $\mathcal{I}^\perp \in l^\infty(L^\infty)$ then the question is the reflexivity of $\mathcal{I}^\perp$ or (equivalently) $(\mathcal{I}^\perp)'$. The latter can be identified with the quotient

$$ba/\overline{span\mathcal{I}}^{\sigma(ba, L^\infty)}.$$

Is it possible to find a reflexive quotient of a non-reflexive Banach space ?

There is another possible approach to this problem: considering a family of independent identically distributed random variables $X_1, X_2, \ldots$, taking investments of the form

$$\Phi = \left(\phi_0, \phi_1(X_1), \ldots, \phi_{T(\Phi)}(X_1, \ldots, X_{T(\Phi)})\right).$$

Then stationarity assumption is that for all $k \in I\!N$ $T^k(\mathcal{I}) \subset \mathcal{I}$, where

$$T^k(\Phi) = I_{\{X_1=k\}} \left(0, \phi_0, \phi_1(X_2), \ldots, \phi_{T(\Phi)}(X_2, \ldots, X_{T(\Phi)+1})\right).$$

Then the adjoint is defined via

$$T^{*k}(H) = (h_1(k), h_2(k, X_1), \ldots) P(X_1 = k).$$

The problem is to find a common fixed point to the family T^{*k}. If it is possible then we find a deflator of the form

$$c\alpha(X_1) \ldots \alpha(X_t)$$

at time t.

6 Measures of density with minimal L^2-norm

We take Bernoulli random variables ε_i and asset returns

$$R_i := \varepsilon_i - b_i,$$

where $-1 < b_i < 1$.

We are looking for the martingale measure Q_n for $R_1, \ldots, R_n$ such that

$$E(\frac{dQ_n}{dI\!P})^2$$

is minimal. We have found that the minimal measure *does not* make the ε_i independent and that the minimal value of the L^2 norm is of the order

$$\sum_{i=1}^{n} b_i^2,$$

in accordance with our intuition.

7 Can we associate to each sequence b_i in ℓ^r a martingale measure with density in $L^{r'}$?

We consider a one-period market model with countably many assets, whoses returns $R^i := \varepsilon_i - b_i$ $i \geq 1$. We suppose that the ε_i are independent with $\varepsilon_i \in L^r$. We furthermore assume ...

We investigate the following implication

$$\sum_{i=1}^{\infty} b_i^r < \infty \Longrightarrow \exists Q \sim I\!P \ \frac{dQ}{dI\!P} \in L^{r'} \text{such that } Q \text{ is EMM.} \tag{7.5}$$

We define the mapping

$$\Psi : L^{r'} \to I\!R^{I\!N}, \quad \phi \to (E\phi\varepsilon_i)_{i\geq 1}.$$

Our problem is equivalent to

$$\Psi(B \cap L_+^{r'}) = \ell^r,$$

where B is the ball of unity in L^1.

Let us consider the case $r = 2$. The first idea is to consider

$$\phi = 1 + \sum_{i=1}^{\infty} b_i \varepsilon_i.$$

ϕ is in L^2, $E\phi = 1$ and $E\phi\varepsilon_i = b_i$, the problem is that ϕ is not necessarily positive.

In the case where ε_i form a complete orthonormal system such that for each n the support of $(\varepsilon_1, \ldots, \varepsilon_n)$ is $I\!R^n$. Such a system exists by results of M. Rásonyi: Arbitrage pricing theory and risk-neutral measures. For all not identically zero sequences b_i there exists a real constant λ such that for λb_i there is no $dQ/dI\!P$ satisfying (7.5). Let us suppose indirectly that for all λ there is $f^\lambda \in B \cap L^2$ such that its Fourier-coefficients are λb_i. By completeness of the system, we have that

$$f^\lambda = 1 + \sum_{i=1}^{\infty} \lambda b_i \varepsilon_i,$$

where the sum converges in L^2. We define

$$g := \sum_{i=1}^{\infty} b_i \varepsilon_i.$$

If the b_i are not identically 0 then $g \neq 0$, for example $P(g < 0) > 0$, we have that for λ large enough $1 + g\lambda$ is negative with positive probability, a contradiction with the fact that the f^λ are densities. The condition of the support of $(\varepsilon_1, \ldots, \varepsilon_n)$ guarantees that, however, there are martingale measures for finite submarkets, even if λ is huge.

Looking at the mapping Ψ one naturally asks the question: how is it possible that the image of $B \cap L_+^{r'}$ under the linear operator Ψ is the whole space ℓ^r ? For this it is necessary (”en gros”) that the eigenvalues tend to infinity. We looked at finite dimensional examples, how ”in the limit” linear images of cones may become subspaces.

Another related idea: assume that $span\{\varepsilon_i - b_i\}^\perp \cap L_+^{r'} = \{0\}$. By a separation theorem we get f_i such that $f_i \in span\{\varepsilon_i - b_i\}^{\perp\perp}$, and $f_i \geq 0$. The biorthogonal is just the span of $\{\varepsilon_i - b_i\}$, so there exists $\lambda \in I\!R$ such that $f_i = \lambda(\varepsilon_i - b_i)$ which contradicts the fact $P(\varepsilon_i - b_i > 0)$, $P(\varepsilon_i - b_i < 0) > 0$.

One deduces that there exists $\phi_i \geq 0$, $\phi_i \neq 0$ such that $E\phi_i(\varepsilon_i - b_i) = 0$. To construct a probability measure for the whole market we have to consider the convergence of

$$\prod_{i=1}^{n} \frac{\phi_i}{E\phi_i}.$$

There is no reason that this product should converge so we should add a normalization condition on the set $span\{\varepsilon_i - b_i\}^\perp$ in order to obtain some condition on each ϕ_i.

An other idea is to consider $span\{\varepsilon_i - b_i,\ i \in I\!N\}^\perp$, but then the separation argument is not sufficient to obtain a contradiction, we also have to assume a no free lunch condition.

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